ChronaQ Foundations I: Finite Survivor-Lineage Entropy and Relational Order in Two-Boundary Quantum Histories

Working paper

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Abstract

ChronaQ asks how an internally ordered observable history can arise when the complete quantum description has no preferred external time and admissibility is fixed by two boundaries. This paper gives a finite answer for systems with stable accumulated records. A fixed two-boundary-admissible history carrier is mapped to successively resolved record prefixes. Genuine refinement supplies a later-to-earlier truncation map, and continuation makes that map surjective. The number of surviving record lineages, and therefore its Hartley entropy, cannot decrease. When the rendered record alternatives also carry a positive refinement-consistent probability law, Shannon's chain rule gives a second monotone: the entropy of the accumulated record increases by the nonnegative conditional entropy of the newly resolved record. These results are assembled as CQ1. Survivor-lineage volume provides its structural ontology anchor, while normalized Shannon entropy is selected as the standard dimensionless coordinate for downstream calculations because it is continuous, has an expected-surprisal interpretation, and composes by the chain rule. A six-record two-boundary model exhibits strict growth, unequal information increments, a future-boundary plateau, and an exact clock-labelled history-state embedding. A second exact model asks which positive-looking refinements may enter CQ1: five candidate rereadings pass support and diagonal-Shannon tests, but A3 decoherence and record repeatability uniquely retain the stable pointer family relative to a declared earlier record. Without refitting, that family then gives phase-invariant probabilities and an exact ln(2) information increment on four frozen future-boundary conditions, while the complementary eraser context remains phase-sensitive. The construction therefore identifies both a relational order and a quantum admissibility gate without assigning a background time coordinate. It also separates survivor volume and record information from probability mass, geometric volume, and thermodynamic entropy, thereby providing a precise first interface for the later ChronaQ Foundations papers.

Keywords: two-boundary quantum mechanics; decoherent histories; relational time; records; Hartley entropy; Shannon entropy; Rényi entropy; Page–Wootters mechanism; ChronaQ

1. Introduction

1.1 The problem of order without external time

Observed physics is organized by records. A detector stores an outcome, a laboratory notebook preserves a sequence, and a clock correlates changes in one subsystem with changes in another. Such records support the distinction between earlier and later even when a fundamental description does not supply an external parameter with that meaning. Canonical quantum cosmology makes the tension especially clear: a global constraint may describe the whole system, while temporal descriptions arise only through correlations among internal degrees of freedom (DeWitt, 1967; Page & Wootters, 1983). Time-neutral quantum mechanics adds a second complication. When both initial and final conditions are allowed, the admissible histories and their probabilities are globally conditioned rather than generated solely from one temporal end (Aharonov et al., 1964; Gell-Mann & Hartle, 1990; Hartle, 2020).

ChronaQ treats these features as premises rather than anomalies. Its first three axioms posit a timeless complete description, two-boundary admissibility, and a positive observable rendering for decoherent record alternatives. The immediate question is then mathematical: what quantity can be proved to move monotonically as an internal observer resolves more of an accumulated record? A useful answer must survive nonuniform probabilities, must not presuppose a spacetime metric, and must distinguish a direction of record refinement from a duration measured in seconds.

The central proposal of this paper is CQ1, a layered finite result. Its structural layer counts the nonempty record lineages that remain compatible with both boundaries. Under genuine refinement and continuation, that survivor volume cannot decrease. Its weighted layer places a positive, consistent probability law on those record prefixes. Shannon entropy then cannot decrease because the entropy of an accumulated record satisfies the chain rule. Its export layer normalizes the Shannon values over a declared clock-bearing interval to give a dimensionless coordinate. These layers answer related but different questions: how many record alternatives survive, how information is distributed among them, and how a convenient scalar chart is assigned to their order.

This separation is essential. Probability mass is conserved when a parent alternative is divided among children. The probability of any selected child is therefore no greater than that of its parent, so the logarithm of selected branch weight is nonincreasing. A completion count behaves differently: the number of nonempty distinguishable children can exceed the number of parents. A monotonicity argument that defines the first quantity and proves a fact about the second changes mathematical type midstream. CQ1 avoids that error by naming survivor volume, Hartley entropy, Shannon entropy, and path surprisal independently.

1.2 The Born-style strategy

The method is deliberately axiomatic. ChronaQ does not begin P1 by deriving its global ontology from a more familiar time-evolution law. It assumes A1–A4 and asks what follows. This is analogous in spirit, though not in content, to the way a probability rule can turn structural assumptions into quantitative predictions: the value lies in exposing exactly which premise does which inferential work. Here A1 makes the target relational, A2 fixes the globally admissible carrier, and A3 licenses ordinary probabilities for the rendered record alternatives. Those three axioms, together with finite positive refinement conditions stated below, support CQ1. A4 is retained as part of the full ChronaQ framework but does not enter the P1 proof.

The result is not a derivation of the Born or Aharonov–Bergmann–Lebowitz (ABL) rules. Rather, A3 declares the domain in which a positive probability law is available after decoherence or stable record formation. The structural support theorem can be proved before probabilities are introduced. The Shannon theorem begins only after the positive law and its refinement consistency have been supplied. This division makes it possible to preserve richer signed or complex latent structures elsewhere in ChronaQ without treating them as probabilities before an observable positive chart has been constructed.

1.3 Contributions

The paper makes seven contributions.

1. It states the four ChronaQ axioms in a common form and gives an explicit premise map for P1.
2. It defines a fixed finite carrier of two-boundary-admissible histories, accumulated record maps, survivor supports, truncation maps, and consistent prefix laws.
3. It proves the structural CQ1 theorem: survivor volume and Hartley entropy cannot decrease under genuine refinement with continuation.
4. It proves the weighted CQ1 theorem: accumulated-record Shannon entropy increases by a nonnegative conditional entropy, and positive path surprisal is nondecreasing as a corollary.
5. It shows that classical Rényi prefix entropies share the refinement monotonicity. Monotonicity therefore does not uniquely select Shannon; Shannon is chosen for the standard coordinate for additional, declared reasons.
6. It supplies an exact six-record calculation and a kinematic clock-labelled history-state embedding.
7. It gives a frozen finite quantum control in which A3 decoherence and record repeatability select one CQ1-eligible lineage from five classically monotone candidates, followed by four no-refit future-boundary consequences.

The general ingredients—two-boundary conditioning, decoherent histories, record-based descriptions, partition entropy, and relational time—are established. The support and Shannon inequalities proved here are standard consequences of finite partition refinement and deterministic coarse-graining (Cover & Thomas, 2006; Marshall et al., 2011; Simovici & Jaroszewicz, 2002). The intended contribution is their disciplined assembly within ChronaQ: a survivor-record interpretation of the partition order in a fixed two-boundary carrier, a separately typed weighted coordinate, an A3 admissibility test that prevents incompatible rereadings from masquerading as record refinements, explicit failure gates, and a downstream interface that does not silently promote record information into geometry or dynamics.

1.4 Organization and status language

Section 2 relates the construction to time-neutral histories, records, and relational clocks. Section 3 states A1–A4. Section 4 defines the finite model. Section 5 gives the three layers of CQ1 and the path-surprisal corollary. Section 6 distinguishes trace weight from survivor volume. Section 7 presents the exact six-record model and controls. Section 8 tests quantum record-family selection beyond positive partition refinement. Section 9 gives a clock-labelled history-state embedding and its precise dynamical limitations. Sections 10 and 11 discuss interpretation, downstream interfaces, limitations, and falsifiers. Appendix A supplies proof details and the Rényi proposition.

Four status terms are used throughout. *Chosen* identifies an axiom, definition, convention, or chart selection. *Forced* identifies a conclusion whose contrary is excluded by an argument or counterexample under stated assumptions. *Conditional* identifies a theorem or calculation valid on a declared domain. *Conjectural* identifies a proposed broader physical reading for which a discriminator or derivation remains open. The ChronaQ axioms are chosen premises; the sign of a conserved child-weight increment is forced; the finite CQ1 results are conditional theorems; and a universal physical arrow remains conjectural.

2. Background and relation to established approaches

2.1 Two-boundary and time-neutral quantum mechanics

The ABL rule conditions the probability of an intermediate outcome on both a preparation and a later selection (Aharonov et al., 1964). Time-neutral decoherent-histories formulations express the same broad possibility at the level of histories by including initial and final density operators in a decoherence functional (Gell-Mann & Hartle, 1990; Hartle, 2020). These frameworks establish that two-boundary probabilities are mathematically coherent in suitable domains. They do not select a unique cosmological final condition, and they do not by themselves prove a monotone record quantity.

The distinction between formal boundary conditioning and operational backward signalling is important. A final boundary can restrict the set of complete histories without becoming a message accessible to an experimenter at an earlier record. The observable probabilities are those defined by the complete conditioning and the permitted measurement context. Any proposed physical realization must still satisfy ordinary operational consistency and no-signalling conditions. CQ1 uses only the resulting finite admissible carrier and its record projections; it does not assume that an observer can manipulate the final boundary.

Time-neutral cosmology also warns against inferring a universal arrow merely from the presence of two boundaries. The orientation and strength of an arrow depend on the boundary conditions, the histories that decohere, and the records used to describe them (Hartle, 2020). A final condition proportional to the identity has a special indifference role, whereas a more selective final condition can alter admissibility and decoherence. ChronaQ must therefore state its ordering construction explicitly. The theorem below does not say that every boundary pair has a nonempty record family; it says what follows when a declared pair does.

2.2 Decoherent histories and stable records

Consistent- and decoherent-histories quantum mechanics represents alternative histories by chains of projections or more general class operators. A decoherence functional determines when interference between distinct alternatives is sufficiently absent for the diagonal terms to obey probability sum rules (Dowker & Halliwell, 1992; Gell-Mann & Hartle, 1993; Griffiths, 1984, 1996; Hartle, 1995; Omnès, 1992). Environment-induced decoherence explains how preferred, robust records can emerge in concrete open-system settings (Joos & Zeh, 1985; Zeh, 1970; Zurek, 1981, 1982, 2003).

Records matter because an ordered history is operationally available only through present correlations. A fine-grained path that leaves no stable distinction cannot serve as a clock reading merely because it appears in a formal sum. Records-based formulations consequently relate histories to alternatives stored in correlated degrees of freedom (Hartle, 2012, 2016). ChronaQ adopts this operational emphasis. Its finite survivor sets consist of nonempty record values induced by admissible complete histories, not of every formal path in an unconstrained expansion.

Decoherence and refinement are separate conditions. Decoherence licenses a positive additive law on a chosen alternative set. Refinement says that a later accumulated record preserves the earlier record and possibly adds a distinction. A family may decohere but intentionally erase an older label; that family is not a refinement in the sense needed here. Conversely, an abstract finite partition may refine perfectly even if no physical mechanism has been shown to stabilize its cells. P1 requires both the structural relation and the A3 rendering domain when the weighted theorem is invoked.

2.3 Relational time and clock-labelled history states

Page and Wootters (1983) showed how a globally constrained clock–system state can encode change relative to a clock subsystem. In the full mechanism, conditioning a solution of a global constraint on a clock reading yields a corresponding state of the remaining system. Modern treatments make the required constraint, reduction map, clock observable, and clock quality explicit, and show that clock–system interactions can modify the conditional dynamics (Höhn et al., 2021; Smith & Ahmadi, 2019). Optical demonstrations have illustrated conditional change and multitime correlations (Moreva et al., 2014, 2017).

P1 uses only a weaker kinematic construction. Any finite normalized family of record states can be placed in a clock-labelled history vector and recovered by projection on the clock label. That algebraic embedding does not establish that the vector solves a Wheeler–DeWitt-type constraint, identify clock and system Hamiltonians, produce unitary relations between conditional states, or derive a Schrödinger equation. It is retained because it displays the CQ1 record family in familiar relational notation, not because it supplies the missing dynamics. The preselected and postselected ensembles used later are conventional two-boundary quantum contexts of the kind organized by the two-state-vector formalism; the ChronaQ addition is the typed record-refinement and export criterion rather than a replacement for that formalism (Aharonov & Vaidman, 2001).

A clock label alone also does not prove an information monotone. A labelled family can have constant supports, erase distinctions, or lack the required refinement relation. Conversely, accumulated records can be ordered before any Hilbert-space clock is introduced. Survivor refinement therefore supplies the order, Shannon entropy supplies a chosen scalar chart, and Section 9 supplies only a kinematic compatibility check.

2.4 Information quantities and their types

Several logarithmic quantities can appear similar in prose while answering different questions.

History-level entropy has an established literature of its own, including analyses of classical-history descriptions and their coarse-grainings (Brun & Hartle, 1999). CQ1 uses a narrower accumulated-record construction and therefore does not identify its monotones with every entropy assigned to a history ensemble.

- **Survivor volume** V_k is the number of nonempty accumulated-record values at level k. It is a dimensionless cardinality.

- **Hartley entropy** $S_0(k)=\ln V_k$ is the logarithm of that support size. It ignores unequal weights and is the structural CQ1 monotone (Hartley, 1928).
- **Shannon entropy** $S_1(k)$ is the expected surprisal of the accumulated record under a positive law. It depends on both support and weights and obeys an exact chain rule (Shannon, 1948a, 1948b).
- **Rényi entropy** H_α is a family that includes Hartley and Shannon as limiting cases and changes its sensitivity to concentration with α (Rényi, 1961).
- **Path surprisal** $I_h(k)$ is minus the logarithm of one positive prefix probability along a realized or selected history.
- **Trace or Born weight** is probability mass assigned to an alternative in a licensed positive quantum chart.
- **Von Neumann entropy** is an operator quantity and is not interchangeable with any of the preceding classical record entropies.
- **Thermodynamic entropy** requires a physical macrostate prescription, conserved quantities, coarse-graining, and an account of work, heat, or statistical typicality.
- **Geometric volume** requires an independently constructed metric or measure.

Only the first four classical record quantities are involved in CQ1. The word *volume* refers to survivor-lineage cardinality, not spatial volume. The word *entropy* always carries a subscript or explicit definition. This typing prevents a mathematical monotone from acquiring physical dimensions or dynamical content by terminology alone.

Table 1 summarizes how the principal antecedents are used.

Table 1
Established frameworks and their role in ChronaQ Foundations I

Framework or quantity	Established role	Role in this paper	Not inferred here
ABL and time-neutral histories	Probabilities conditioned by two boundaries	Conventional positive comparison and motivation	A unique cosmological final boundary
Decoherent histories	Additive probabilities for consistent alternatives	Domain for A3 record probabilities	Permanent branching at every scale
Records formulations	Correlations that make history alternatives available	Defines the operational refinement objects	Classical reality for every latent path
Page–Wootters and history-state formalisms	Conditional change from an appropriately constrained clock–system state	Kinematic embedding of the finite record family	A global constraint, Hamiltonians, unitary dynamics, clock quality, rate, or duration
Hartley entropy	Logarithm of support cardinality	Structural survivor-volume monotone	Probability mass or geometric volume
Shannon entropy	Expected information with a chain rule	Weighted monotone and standard coordinate	Unique physical entropy or local production
Rényi family	Alternative classical entropy orders	Demonstrates non-uniqueness of monotonicity	A preferred clock coordinate
Trace or Born weight	Positive observable probability	Retained as probability; supplies exclusion	Increasing completion volume

3. The ChronaQ axioms and their role in P1

3.1 A1: timeless totality

A1. The complete description is timeless: the total quantum object has no distinguished external time parameter. In a constrained Hilbert-space representation, timelessness may be expressed schematically in Wheeler–DeWitt form as

$$H|\Psi\rangle=0 \tag{1}$$

where the constraint operator is not a Hamiltonian generating evolution in external time (DeWitt, 1967). This notation is a conventional realization of A1, not the only possible one. The finite theorem below does not require a particular constraint operator. It requires that the labels (k) initially be treated as levels of accumulated record resolution rather than as pre-existing instants.

A1 determines the kind of result sought. If external time were already primitive, one could simply ask whether an entropy changes with that parameter. Under A1, order must instead be constructed from relations available within the complete description. Accumulated records are suitable because a later resolved record can contain an earlier prefix without appealing to an external clock.

3.2 A2: two-boundary admissibility

A2. Complete histories are admissible only when they satisfy both boundary conditions. Let the boundaries be denoted B_- and B_+. In a histories chart, a coarse-grained alternative α may be represented by a class operator

$$C_\alpha=P_{\alpha_n}(k_n)\cdots P_{\alpha_1}(k_1) \tag{2}$$

and a time-neutral decoherence functional may be written

$$D_{-+}(\alpha,\beta)=\frac{Tr[\rho_+C_\alpha\rho_-C_\beta^{\dagger}]}{Tr[\rho_+\rho_-]} \tag{3}$$

when the denominator is nonzero and the family meets the required consistency conditions. The formula is a conventional positive chart for two-boundary histories. A2 itself is more general: it says that the admissible whole is fixed jointly rather than generated from only one end.

At the finite level, let Ω be a candidate set of complete microhistories. The fixed two-boundary-admissible carrier is

$$\Omega_{B_-,B_+}=\{h\in\Omega:B_-(h)=1\wedge B_+(h)=1\} \tag{4}$$

The subscript is important. It records that the carrier has already been filtered by both boundaries before record prefixes are compared. P1 assumes that this set is finite and nonempty. An empty carrier is an exclusion of the proposed boundary/model combination, not an invitation to renormalize or change the boundaries after inspecting the result.

3.3 A3: positive observable rendering

A3. Observable alternatives are rendered through positive, normalized probabilities when their interference is absent or controlled and stable records distinguish them. For a finite exact history family, a sufficient diagonal condition is

$$D_{-+}(\alpha,\beta)=0(\alpha\neq\beta) \tag{5}$$

for α≠β. More generally, an approximate calculation must declare a decoherence tolerance and show that its conclusions are stable under that tolerance. Once the A3 gate is met, the diagonal weights can be treated as an ordinary probability law for the rendered alternatives.

A3 is deliberately a rendering axiom, not a statement that every latent object in ChronaQ is a positive measure. A latent representation may retain phase, signed components, or several compatible completions. Such objects cannot enter Shannon entropy as probabilities until a positive observable chart has been constructed. P1 therefore uses A3 twice: first to justify record alternatives as operationally meaningful, and second to license the positive prefix law used in the weighted theorem. P1 does not derive the Born or ABL rule from more primitive assumptions.

3.4 A4: entanglement–geometry compatibility

A4. Relational quantum structure constrains a family of compatible geometric descriptions. Entanglement or other relational data may therefore provide inputs to a later geometry interface, but a metric, signature, scale, and continuum limit require their own admissibility conditions. A4 is an assumption of the full ChronaQ programme. It is not a universal formula equating distance with the reciprocal of a single information quantity.

A4 has no inferential role in P1. Neither the survivor-support proof nor the Shannon chain rule uses a metric, mutual information, causal cone, coframe, connection, or action. The separation is constructive: P1 supplies an order that later work may seek to represent geometrically, while A4 tells that later work what sort of compatibility question to ask. It does not allow geometry to be read directly from S_0, S_1, or τ.

3.5 Premise map

The logical division is compact.

Table 2
Premise map for the CQ1 construction

Premise	P1 role	Consequence supported
A1	Removes primitive external time; makes record order relational	(k) is initially a refinement index
A2	Fixes the admissible complete-history carrier using both boundaries	Survivor sets are images of one globally conditioned carrier
A3	Licenses stable records and a positive consistent prefix law	Shannon and path-surprisal statements
A4	Declares a later compatibility interface	No step of the CQ1 proof
Finite genuine refinement	Supplies single-parent truncation and continuation	Hartley survivor monotonicity
Positive pushforward consistency	Makes successive prefix laws marginals of one law	Shannon chain-rule monotonicity

Thus CQ1 is not obtained from the labels A1–A3 alone. It follows from those ChronaQ premises plus explicit finite-domain assumptions about the record family. Keeping those assumptions visible makes the theorem falsifiable in applications.

4. Finite two-boundary record model

4.1 Complete histories and accumulated records

Fix a finite nonempty carrier Ω_{B_-,B_+}. Each element $h \in \Omega_{B_-,B_+}$ is a complete microhistory compatible with both boundary conditions. No probability need yet be assigned. At resolution level k, let $Z_{\leq k}=(Z_0,\ldots,Z_k)$ denote the accumulated record, and let

$$r_k : \Omega_{B_-,B_+} \to R_k \tag{6}$$

map a complete history to its accumulated record value. The codomain R_k may contain formally possible values that no admissible history realizes. The survivor support is the image

$$L_k = r_k(\Omega_{B_-,B_+}) \subseteq R_k \tag{7}$$

so every element of L_k has at least one two-boundary-admissible completion. Two complete histories belong to the same lineage at level k when they produce the same accumulated record. The fibres of r_k therefore partition the carrier into operationally unresolved equivalence classes.

This definition separates microhistory multiplicity from rendered record multiplicity. Many microhistories may share one accumulated record and hence contribute only one element to L_k. Conversely, one earlier record may split into several later record values when a new stable distinction is resolved. Survivor volume counts the record lineages, not the cardinality of the hidden microhistory carrier.

4.2 Genuine refinement and truncation

A later accumulated record genuinely refines the earlier one when forgetting the newly resolved component recovers a unique earlier prefix. Formally there is a truncation map t_k satisfying

$$r_k = t_k \circ r_{k+1}, t_k : L_{k+1} \to L_k \tag{8}$$

The factorization encodes record preservation. Every fine record has one parent. Several fine records may share that parent, which is exactly how a refinement creates new distinctions. A later label that merges parts of two earlier record cells cannot satisfy this factorization and lies outside the theorem's domain.

For a fixed carrier the map t_k:$L_{k+1} \to L_k$ is automatically surjective. Given any $\ell \in L_k$, some complete history h realizes it. The later record $r_{k+1}(h)$ exists on the same complete history and truncates to ℓ. In a model that recomputes or prunes the carrier at each level, this argument no longer follows automatically; continuation must then be checked as an independent gate.

The physical interpretation of continuation is modest but important. A record that is declared viable at level k cannot disappear merely because a finer description is opened. It may acquire several continuations or only one. If a later boundary filter eliminates it, then the earlier set was not the image of the same fixed two-boundary carrier, and the proposed application must be analysed as a changing-carrier problem.

4.3 Prefix laws

For the weighted layer, let $q(h) \geq 0$ be a normalized probability law on the admissible carrier, or equivalently let a positive law be defined on the complete rendered records. The accumulated-record law is its pushforward:

$$p_k(z_{\leq k}) = \sum_{h: r_k(h) = z_{\leq k}} q(h) \tag{9}$$

Only values in L_k have positive mass. Refinement consistency means that the coarser law is the pushforward of the finer law under t_k. In ordinary notation, parent probability is the sum of its children. The successive p_k are therefore marginals of one globally conditioned law, not independently fitted distributions.

Strict positivity on the reported support avoids undefined logarithms. Zero-weight formal alternatives are omitted from L_k for the weighted calculation. This convention does not force all candidate microhistories to have positive weight; it says only that the rendered support contains the alternatives assigned positive probability.

4.4 Four derived quantities

Define survivor volume and Hartley entropy by

$$V_k = |L_k|, S_0(k) = \ln V_k = \ln |L_k| \tag{10}$$

Define accumulated-record Shannon entropy by

$$S_1(k) = -\sum_{z_{\leq k}} p(z_{\leq k} | B_-, B_+) \ln p(z_{\leq k} | B_-, B_+) \tag{11}$$

For a positive realized or selected path h, define its prefix weight $W_h(k)=p_k(r_k(h))$ and path surprisal $I_h(k)=-\ln W_h(k)$. Finally, once two ordered clock-bearing reference levels have a positive Shannon span, define the standard dimensionless coordinate by

$$\tau_k=\frac{S_1(k)-S_1(k_0)}{S_1(k_*)-S_1(k_0)} \tag{12}$$

The first two quantities describe an ensemble of alternatives; the third describes one positive path; the fourth is a chart constructed from the Shannon ensemble values. Their dimensions are all unity. None is a duration until a separate operational clock calibration is supplied.

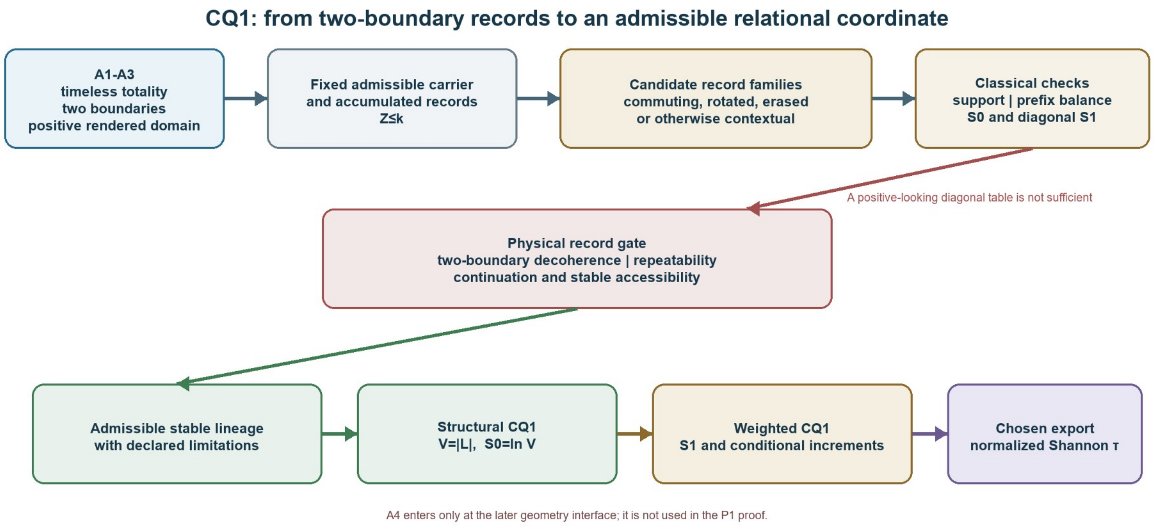

Figure 1. CQ1 construction. A1 removes external time, A2 fixes the two-boundary-admissible carrier, and A3 licenses stable positive records. Genuine refinement yields structural Hartley and weighted Shannon monotones; normalized Shannon entropy is exported as the standard dimensionless coordinate. A4 enters only at the later geometry interface.

5. CQ1: structural order, weighted order, and clock export

5.1 Theorem 1: structural survivor-volume monotonicity

Theorem 1 (finite survivor refinement). Let Ω_{B_-,B_+} be finite and nonempty. Let r_k and r_{k+1} be accumulated-record maps whose survivor supports are related by a truncation t_k such that $r_k=t_k\circ r_{k+1}$. Then

$$|L_{k+1}|\geq|L_k|, S_0(k+1)\geq S_0(k) \tag{13}$$

Proof. The factorization on a fixed carrier makes $t_k:L_{k+1}\to L_k$ surjective. A surjection between finite sets has a domain whose cardinality is at least that of its codomain. The natural logarithm is increasing, so the same order holds for S_0. ■

The theorem supplies CQ1's ontology anchor. A survivor lineage is not probability mass. It is a nonempty class of complete histories that share an accumulated record. The cardinality increases when resolving a new record splits at least one parent class; it remains constant when every parent has exactly one surviving child.

The child-count form makes the strictness condition explicit. Let

$$|L_{k+1}|=\sum_{\ell\in L_k} n_k(\ell), n_k(\ell)=|t_k^{-1}(\{\ell\})|\geq 1 \tag{14}$$

Then equality holds exactly when every parent has one child. Strict growth occurs exactly when at least one parent has more than one child. A refinement index can therefore advance while the structural entropy plateaus. Such an index change is not automatically a clock tick.

5.2 Theorem 2: weighted Shannon monotonicity

Theorem 2 (finite accumulated-record entropy). In addition to the assumptions of Theorem 1, suppose the rendered survivor records carry positive laws p_k and p_{k+1} related by pushforward under truncation. Then

$$S_1(k+1)-S_1(k)=H(Z_{k+1}|Z_{\leq k},B_-,B_+)\geq 0 \tag{15}$$

Proof. Since $Z_{\leq k}$ is a deterministic function of $Z_{\leq k+1}$, the joint random variable $(Z_{\leq k},Z_{k+1})$ is equivalent to $Z_{\leq k+1}$. Shannon's chain rule gives the displayed identity. Classical conditional entropy is nonnegative. ■

The increment is zero precisely when the newly appended record is determined with conditional probability one for every positive parent. It is positive when at least one parent has more than one positive-probability child. Unlike Hartley entropy, the size of the Shannon increment depends on how weight is distributed among those children. A nearly deterministic split can create a new support point while adding very little Shannon information.

The chain rule is particularly useful for a relational coordinate because it localizes each finite increment without reinterpreting it as a spacetime density. One can say exactly how much expected record information is added at the next refinement. One cannot yet say where in a continuum that information is produced or which current transports it.

5.3 The standard Shannon coordinate

When the ending reference level has greater Shannon entropy than the starting reference level, Equation (12) maps the declared clock-bearing interval to $0\leq\tau_k\leq 1$. This normalization removes an arbitrary additive origin and positive multiplicative scale. The resulting coordinate preserves the Shannon order, and unequal increments preserve information about how strongly each record step resolves the positive law.

Shannon entropy is *chosen* as ChronaQ's standard CQ1 chart coordinate. It is not uniquely forced by monotonicity. The choice rests on four properties used downstream.

- The chain rule decomposes the total accumulated information into conditional increments.
- Continuity makes the coordinate less brittle than pure support count when small positive weights vary.
- Shannon entropy is the expected value of path surprisal, giving an immediate ensemble interpretation.
- Existing rendered calculations in the ChronaQ action programme already use the normalized Shannon coordinate, so retaining it preserves a stable interface.

These reasons select a standard chart without claiming a unique physical entropy. Other monotone reparameterizations preserve the same order, and other classical entropies may be appropriate diagnostic coordinates. The chart is undefined when the Shannon span is zero. A deterministic record family may possess a perfectly valid survivor order while supplying no Shannon clock coordinate.

5.4 Rényi non-uniqueness

For $\alpha\geq 0$, classical Rényi entropy is

$$H_\alpha(p)=\frac{1}{1-\alpha}\ln\sum_x p^\alpha(x),\alpha\neq 1 \tag{16}$$

with the usual continuous extension at $\alpha=1$, the Hartley value at $\alpha=0$, and min-entropy as $\alpha\to\infty$. Appendix A proves that every member of this family is nondecreasing when a positive fine distribution is mapped to its coarse parent distribution by deterministic truncation:

$$H_\alpha(p_{k+1})\geq H_\alpha(p_k),\alpha\in[0,\infty] \tag{17}$$

The result matters conceptually. A claim that "entropy increases under refinement" cannot, by itself, identify which entropy should be treated as the temporal coordinate. Hartley emphasizes the existence of alternatives; high-order Rényi entropies emphasize the most probable alternatives; Shannon measures expected surprisal. CQ1 assigns structural priority to Hartley because its support is the survivor-lineage ontology anchor, and it assigns chart priority to Shannon because of the additional properties listed above.

5.5 Corollary: pathwise surprisal

Along any positive path h, refinement consistency factorizes the prefix probability:

$$W_h(k+1)=W_h(k)\,p(z_{k+1}|z_{\le k},B_-,B_+) \tag{18}$$

Since a positive conditional probability lies in (0,1],

$$I_h(k+1)-I_h(k)=-\ln p(z_{k+1}|z_{\le k},B_-,B_+)\ge 0 \tag{19}$$

Thus positive path surprisal is nondecreasing. Its expectation under p_k is $S_1(k)$, but the pathwise and ensemble statements are not interchangeable. A low-probability path can gain a large surprisal increment while contributing little probability weight to the ensemble average.

Path surprisal is retained as a corollary and diagnostic. P1 does not interpret its finite increment as a local entropy density, production rate, energy, action contribution, or elapsed duration. Such an interpretation would require a local domain, a measure, and a dynamical conservation or balance law that are absent from the finite record theorem.

5.6 CQ1 in compact form

CQ1 can now be stated without conflation.

> For a finite nonempty two-boundary-admissible history carrier whose stable accumulated records form a genuine refinement with continuation, survivor volume V_k and Hartley entropy $S_0(k)$ are nondecreasing. If A3 supplies a positive refinement-consistent record law, accumulated Shannon entropy $S_1(k)$ is also nondecreasing by the chain rule. Over any declared positive Shannon span, normalized S_1 defines the standard dimensionless coordinate τ_k. The coordinate is chosen, while the two monotonicity results are conditional theorems.

This compact statement is the result exported to the remainder of the Foundations series.

5.7 Order invariants and chart-dependent statements

The layered form of CQ1 separates statements that survive a change of coordinate from statements that use the specific Shannon spacing. The existence of a truncation map, its surjectivity, the comparison $k \preccurlyeq k+1$, and the distinction between strict refinement and plateau do not depend on a numerical entropy coordinate. Hartley, Shannon, the selected Rényi entropies, and any strictly increasing function of them can all represent the same direction when their increments are positive. These are order-level statements.

Numerical derivatives, finite-difference coefficients, and comparisons of the sizes of successive increments are chart dependent. For example, Table 4 shows that the fourth refinement has a Shannon increment of 0.525963 but a min-entropy increment of 0.101164. Both coordinates agree that a positive refinement occurred; they do not assign it the same separation. A downstream action written using $\Delta\tau$ therefore contains a declared Shannon-chart choice unless it can be shown invariant under admissible reparameterization.

This distinction is familiar in geometry, where coordinates permit calculation without themselves being observables. Here it is imposed before a geometry exists. The partial order carried by record inclusion is the invariant core, while τ is a standard chart on a finite chain or ordered subfamily. If two record refinements are incomparable—for example, if they resolve different records without a common prefix relation—CQ1 does not force them onto one scalar line. A total clock sequence then requires a chosen chain, an additional compatibility rule, or a later theorem.

The distinction also prevents circular reasoning. One cannot define "later" as larger Shannon entropy and then claim that Shannon entropy increases with time. Later means finer accumulated record under the stated truncation relation. The entropy theorem is a consequence of that relation. The normalized coordinate is adopted only after the direction and positive span have been established.

6. Why trace weight is not survivor volume

6.1 Conservation under refinement

Let a positive parent alternative B be partitioned into mutually exclusive children B_i. Probability additivity gives

$$W_B=\sum_i W_{B_i}, W_{B_i}>0 \tag{20}$$

so every child satisfies $W_{B\_i} \leq W_B$. Consequently,

$$\ln W_{B_i}-\ln W_B\leq 0 \tag{21}$$

The logarithm of selected child probability is nonincreasing. The negative logarithm is nondecreasing. This sign is forced by positive probability conservation, independently of ChronaQ.

A two-child example is sufficient. Split a parent of weight 1 into children of weights 0.6 and 0.4. The selected 0.6 child has log increment ln0.6=-0.5108256. Meanwhile the support grows from one alternative to two, so Hartley entropy increases by ln2. The same refinement therefore makes log branch weight decrease and support entropy increase. There is no contradiction because the quantities have different types.

6.2 Mass, volume, and information

Probability mass answers how likely a rendered alternative is. Survivor volume answers how many nonempty record alternatives remain. Shannon entropy combines both support and mass into an ensemble average. Path surprisal changes the sign of a positive path's log probability. None of these is automatically a spatial volume or a thermodynamic entropy.

The distinction also clarifies what two-boundary conditioning does. Conditioning can change the normalized weights of complete histories and can remove histories from the admissible carrier. After the carrier and record family are fixed, refinement distributes each positive parent weight among its positive children. The ordering theorem cannot be rescued by renaming a decreasing child mass as increasing volume. It is instead grounded in the increasing number of distinguishable surviving record values.

6.3 Relation to Born and ABL probabilities

CQ1 does not challenge Born or ABL probabilities in their licensed domains. It rejects only the inference that normalized probability mass is a generally increasing completion count. A3 treats the positive rendered law as an input for the weighted theorem. Whether that law arises from Born weights, ABL conditioning, a decoherence functional, or another empirically equivalent positive construction is a separate foundational question.

This restraint preserves a useful ChronaQ distinction. The global or latent representation may contain signed or complex structure that is essential for interference and compatibility. Observable probabilities must nevertheless be positive and normalized where Shannon entropy is calculated. A positive local export does not prove a unique global latent measure, and a nonunique latent completion does not prevent a well-defined local record law.

7. Exact six-record model and controls

7.1 Definition of the two-boundary law

Consider six binary records $z_j\in\{-1,+1\}$, indexed by j=0,…,5. The past boundary fixes $z_0=+1$. The future boundary fixes the parity $\Pi_{j=0}^{5} z_j=+1$. On the compatible complete records, define

$$q(z)=N^{-1}\exp[0.45\,z_1 z_4-0.30\,z_2 z_5+0.20\,z_1 z_2+0.15\,z_3 z_4+0.22\,z_1 z_3 z_5] \tag{22}$$

where $\mathcal{N}$ normalizes the law after both boundaries are imposed. The interaction terms make the positive law nonuniform and include pair and triple correlations. The model is globally conditioned. It is not assumed to be a first-order forward Markov process.

Sixteen complete histories satisfy both boundaries. Records z_1,z_2,z_3,z_4 are then accumulated in sequence. The parity constraint determines z_5 after the preceding values are known, so the final index closes each surviving history without creating another independent distinction.

7.2 Exact prefix balance and entropy results

At every node the prefix probability equals the sum of its positive child probabilities:

$$p\left(z_{\leq k}\right)=\sum_{z_{k+1}} p\left(z_{\leq k+1}\right) \tag{23}$$

The maximum balance residual is 5.55×10^{-17}. The support counts, Hartley entropies, Shannon increments, and normalized Shannon coordinate are shown in Table 3.

Table 3
Support and Shannon-coordinate results for the six-record law

Prefix level k	Resolved role	V_k	$S_0(k)$	$S_1(k)-S_1(k-1)$	τ_k
0	Past boundary only	1	0.000000	—	0.000000
1	Reveal (z_1)	2	0.693147	0.692206	0.271318
2	Reveal (z_2)	4	1.386294	0.652372	0.527023
3	Reveal (z_3)	8	2.079442	0.680730	0.793843
4	Reveal (z_4)	16	2.772589	0.525963	1.000000
5	Future parity closes (z_5)	16	2.772589	0.000000	no new tick

The first four refinements strictly increase support and Shannon entropy. Their unequal Shannon increments show why τ is not merely a relabelled integer index. The final closure is a plateau for both S_0 and S_1: the record is conditionally certain and adds no new alternative or expected information. Treating every formal index as one unit of elapsed time would miss this distinction.

For the declared clock-bearing levels 0 through 4, Equation (12) yields

$\tau=(0,0.271318,0.527023,0.793843,1)$.

The coordinate has no unit. A positive affine rescaling would preserve the order, but a conversion to seconds would require an independently calibrated clock mechanism.

Figure 2 shows the support, Shannon, and normalized-clock trajectories together. The future-boundary closure appears as a genuine plateau rather than an additional clock tick.

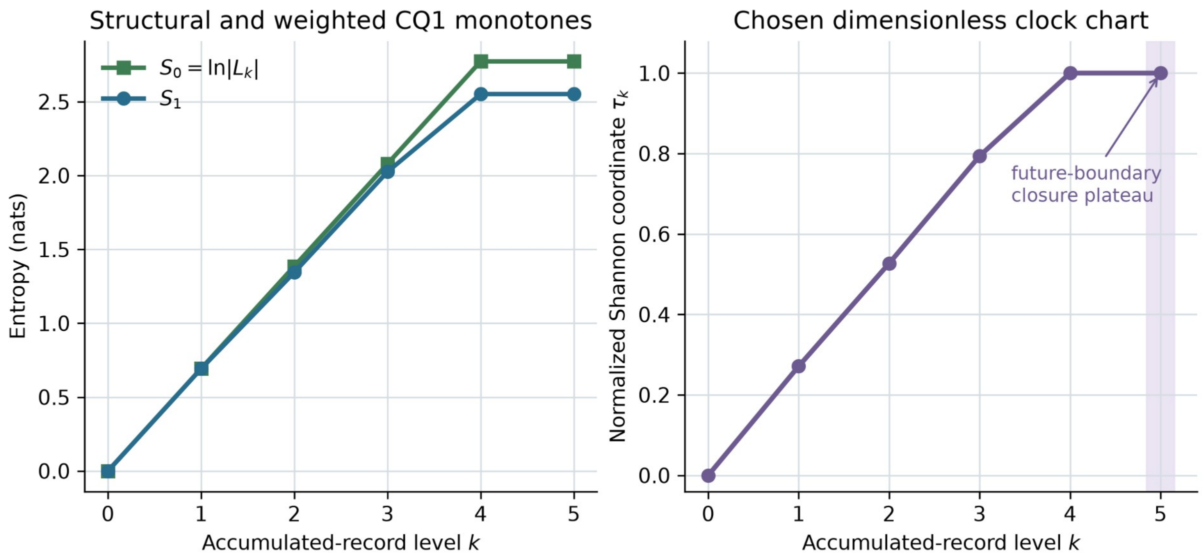

Figure 2. Exact six-record trajectory. Survivor support and Hartley entropy grow through four genuine refinements, Shannon entropy grows by unequal conditional increments, and normalized tau reaches one before the future-boundary closure. The final level is a plateau and supplies no new tick.

7.3 Selected Rényi checks

The same prefix law was evaluated at α=0,0.5,1,2,∞. Every selected entropy is nondecreasing, and every family member has a zero final increment. Table 4 reports the increments.

Table 4
Selected Rényi-entropy increments under positive prefix refinement

α	Increment 0→1	Increment 1→2	Increment 2→3	Increment 3→4	Closure 4→5
0	0.693147	0.693147	0.693147	0.693147	0.000000
0.5	0.692677	0.672403	0.687509	0.609824	0.000000
1	0.692206	0.652372	0.680730	0.525963	0.000000
2	0.691268	0.615783	0.664394	0.378977	0.000000
∞	0.650689	0.443470	0.472565	0.101164	0.000000

The family shares the order but assigns different spacings to it. As α increases, the coordinate becomes more sensitive to the largest probabilities. The calculation therefore confirms both sides of the CQ1 selection statement: monotonicity is robust across a classical entropy family, and monotonicity alone does not privilege Shannon.

7.4 Path-surprisal control

Along the maximum-weight complete history, the prefix weights are

$$W_h = (1, 0.521686, 0.334821, 0.208728, 0.188645, 0.188645) \tag{24}$$

The corresponding log-weight increments are (-0.650689,-0.443470,-0.472565,-0.101164,0). Their negatives are the nonnegative path-surprisal increments predicted by Section 5.5. The model therefore displays three compatible facts: selected prefix probability decreases, selected path surprisal increases, and ensemble Shannon entropy increases until the boundary-determined plateau.

The globally conditioned law also permits the finite diagnostic

$$M_4 = \max_{z_1, z_2, z_3} \left| p(z_4 = +1 \mid z_1, z_2, z_3, B_-, B_+) - p(z_4 = +1 \mid z_3, B_-, B_+) \right| \tag{25}$$

For the declared six-record law, M_4=0.4119774542309629. Thus the first-order Markov conditional-independence relation fails in the declared record ordering: knowledge of (z_1,z_2) can change the conditional probability of z_4 even after z_3 is fixed. This is a property of one globally conditioned classical record distribution. It is not evidence of quantum non-Markovian dynamics, does not define a memory kernel, and is not required by CQ1. Its limited role is to show that the example is not equivalent to a first-order forward transition chain while it still satisfies prefix balance.

7.5 Domain and failure controls

Small controls expose the theorem's gates without reliance on an optimizer.

Table 5
Positive and negative controls for the CQ1 domain

Control	Earlier support	Later support	Gate result	Consequence
Passing refinement	2	3	Surjective single-parent truncation	Structural theorem applies
Missing continuation	2	1	One parent has no child	Outside theorem domain
Invalid merge	2	2	One fine cell is assigned two parents	Not a refinement map
Conserved trace split	parent 1.0	children 0.6, 0.4	Positive weights sum to parent	Child log weight decreases
Deterministic family	1	1	Zero entropy span	Order may exist; τ undefined
Future closure	16	16	New record conditionally certain	Valid plateau; no new tick

The negative controls are part of the scientific content. Missing continuation does not numerically "miss" the theorem; it violates a premise. An invalid merge is not repaired by changing a tolerance because it lacks a single-valued truncation. A deterministic family cannot be given a Shannon clock by division through a zero span. These outcomes should be reported rather than tuned away.

7.6 What the exact model establishes

The calculation verifies both boundary predicates, positive normalization, exact support counts, prefix balance, the declared Shannon increments, selected Rényi monotonicity, the future-boundary plateau, normalized τ, the first-order Markov diagnostic, and the kinematic embedding below. It contains no fitted physical scale, target duration, energy, species label, metric, or gravitational parameter. It is an exact finite existence and consistency model for CQ1, not evidence that the universe realizes this particular six-record law.

8. Quantum record-family selection beyond positive partition refinement

8.1 Why an ordinary partition test is insufficient

The CQ1 inequalities apply to a valid accumulated-record family; they do not make every finer list of positive diagonal numbers into such a family. In quantum mechanics, incompatible histories can have nonnegative normalized diagonal weights while retaining interference in the off-diagonal decoherence functional. A purely classical audit of support and Shannon entropy cannot see that interference. Nor can it determine whether a later label rereads the established record repeatably or replaces it with an incompatible context.

A3 supplies the missing gate. Once an earlier record algebra has been physically declared, a candidate continuation must satisfy the relevant decoherence condition and preserve the record under repetition. This criterion is stronger than ordinary partition refinement but still conventional quantum mechanics: it uses the decoherence functional, projective measurement, and ABL conditioning. Its ChronaQ role is to determine which positive chart may enter CQ1.

8.2 Exact two-boundary control

The control uses two independent Bell-correlated path–marker pairs. The first marker already stores a Z record. Five proposed later rereadings rotate from that pointer basis through angles $\theta/\pi = 0, 0.125, 0.25, 0.375$, and 0.5, ending in the complementary X eraser context. A future boundary on the first path qubit is specified by polar angle β and phase φ. Candidate class operators are evaluated with the time-neutral decoherence functional of Equation 3. The selection values are calculated on three training boundaries; four separately frozen future boundaries are opened only if one candidate identity survives.

This arrangement follows the standard distinction between preselected, postselected, and pre/postselected contexts (Aharonov & Vaidman, 2001). The eraser is therefore not labelled a failed measurement. It is a valid complementary context, but it is not automatically a stable continuation of the earlier Z record.

8.3 Training selection

All five candidate diagonal tables are nonnegative and normalized, satisfy prefix balance, and increase both support and diagonal Shannon entropy. A classical refining-partition audit therefore accepts all five. The quantum record gate does not.

Table 6
Classical and quantum gates for five candidate rereadings

Candidate	θ/π	Classical support/Shannon gate	Maximum normalized off-diagonal defect	Minimum repeatability	Apparent Shannon increment (nats)	CQ1-record eligible
Pointer Z	0.000	Pass	0.000000	1.000000	0.693147	Yes
Rotated 22.5°	0.125	Pass	1.000000	0.961940	0.854877	No
Rotated 45°	0.250	Pass	1.000000	0.853553	1.109643	No
Rotated 67.5°	0.375	Pass	1.000000	0.691342	1.311170	No
Eraser X	0.500	Pass	1.000000	0.500000	1.386294	No

Only the pointer-Z continuation is exactly diagonal and perfectly repeatable relative to the established Z record. The excluded alternatives can exhibit a larger diagonal entropy increment precisely because a noncommuting rereading creates additional positive-looking labels. Histories that differ in the earlier marker value nevertheless retain unit normalized interference. The comparison therefore supplies a selection rule that support cardinality and diagonal entropy alone cannot provide.

8.4 No-refit future-boundary consequences

After the candidate identity was fixed, the selected family was evaluated on two pairs of equal-β, different-phase future boundaries. For the stable record, the earlier weights are $\cos^2(\beta/2)$ and $\sin^2(\beta/2)$: they depend on the polar component of the future boundary but not its phase. Appending the independent second Z record divides each weight equally and adds exactly $\ln(2)$ nats. The directly evaluated X eraser probability remains phase-sensitive, as the conventional ABL control requires.

Table 7

Held-out boundary consequences of the selected record family

Boundary	β/π	φ/π	Stable Z weights	Shannon increment (nats)	Direct eraser weights
Low A	0.31	0.05	(0.781042, 0.218958)	0.693147	(0.908449, 0.091551)
Low B	0.31	0.95	(0.781042, 0.218958)	0.693147	(0.091551, 0.908449)
High A	0.73	0.15	(0.169344, 0.830656)	0.693147	(0.834177, 0.165823)
High B	0.73	1.15	(0.169344, 0.830656)	0.693147	(0.165823, 0.834177)

Within the equal-β pairs, the largest stable-record change is 1.11×10^{-16}. The corresponding eraser-plus probability changes by 0.816898 and 0.668354. Figure 3 shows both the training discriminator and the held-out boundary dependence.

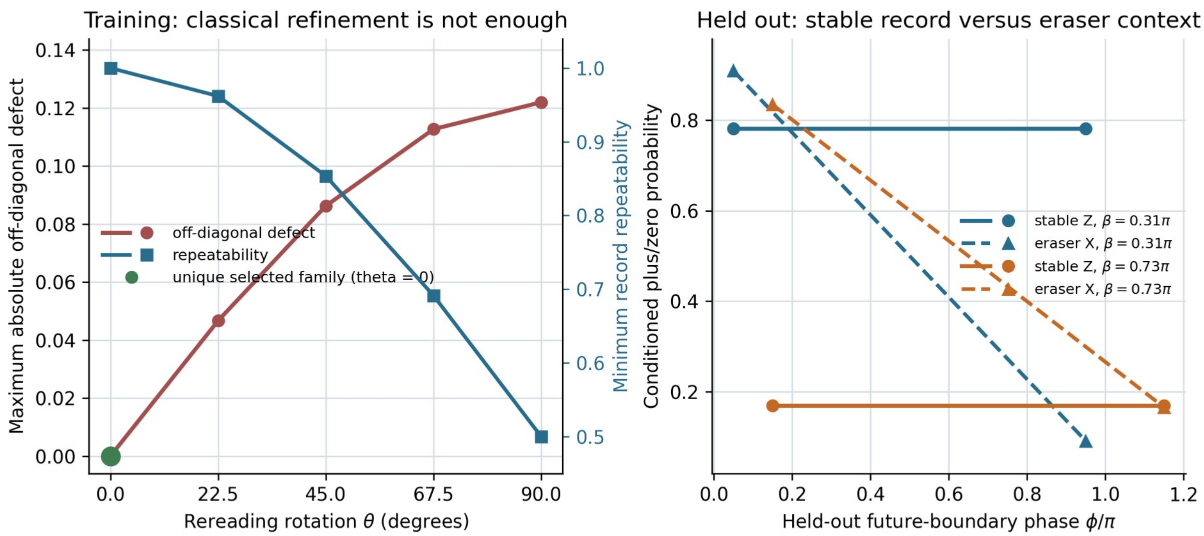

Figure 3. Record-family selection and held-out boundary response. Every candidate has a positive apparent Shannon increment, but only pointer Z has zero normalized off-diagonal defect and unit repeatability. On frozen equal-polar-angle boundary pairs, its weights are phase-invariant while the complementary eraser response changes strongly with phase.

8.5 Scope of the result

The result demonstrates something that an ordinary refining-partition account does not: quantum record conditions select which positive-looking refinements are admissible for CQ1, and that selection carries a frozen boundary-dependent consequence. It does not constitute a new prediction beyond quantum mechanics. Bell correlations, ABL conditioning, decoherence, complementarity, and eraser response are conventional. The ChronaQ content is the typed requirement that a CQ1 export pass A3 and preserve its declared record lineage before its entropy is interpreted as the standard clock coordinate.

The selection is relative to the established Z record algebra. A different record-forming interaction or earlier anchor can require a different basis and a new selection calculation. The result therefore does not derive a universal pointer basis, a cosmological final condition, or a universal thermodynamic arrow.

9. Kinematic clock-labelled history-state embedding

9.1 Construction

Let the clock states $|k\rangle_C$ be orthonormal labels for the five clock-bearing prefix levels, and let the record states $|\psi_k\rangle_S$ encode the square roots of the positive prefix probabilities in an orthonormal record basis. Define the kinematic history vector

$$|\Psi_{\text{kin}}\rangle = \frac{1}{\sqrt{N_C}} \sum_k |k\rangle_C |\psi_k\rangle_S \tag{26}$$

Conditioning on the clock projector gives

$$\rho_S(k) = \frac{Tr_C[(|k\rangle\langle k|_C \otimes I_S)|\Psi\rangle\langle\Psi|]}{\langle\Psi|(|k\rangle\langle k|_C \otimes I_S)|\Psi\rangle} = |\psi_k\rangle\langle\psi_k| \tag{27}$$

and recovers the corresponding system state. In the six-record model the maximum conditional-state residual is 3.33×10^{-16}, while the clock probabilities are uniform to 8.33×10^{-17}. The finite conditional family is therefore embedded and recovered to numerical precision.

9.2 What the representation adds

The construction displays the CQ1 record sequence in the clock-labelled form used by relational-time and history-state approaches. It addresses a modest representational question raised by A1: a family ordered by internal records can be encoded without adding an external time argument to the total vector.

This conclusion is deliberately kinematic. Any finite normalized family $\{|\psi_k\rangle_S\}$ admits the same embedding after a clock basis and clock weights are chosen. P1 has not supplied operators H_C and H_S for which $(H_C + H_S)|\Psi_k\text{in}\rangle = 0$, a physical-state or gauge-reduction map, or unitary relations connecting successive conditional states. It therefore does not establish a stationary Page–Wootters physical state, a Wheeler–DeWitt solution, a Schrödinger evolution, clock accuracy, duration, or energy. Those are additional dynamical and operational obligations (Höhn et al., 2021; Smith & Ahmadi, 2019).

The embedding also does not prove Theorems 1 or 2. Their premises concern refinement and positive marginal consistency. Writing a dimensionful parameter as $t=T_C\tau$ remains only a conditional parametrization until a physical clock and the scale T_C are independently identified.

9.3 Relation to experiment

Optical demonstrations of conditional relational-time descriptions support the operational coherence of that broader framework (Moreva et al., 2014, 2017). They do not upgrade P1's algebraic embedding into a constrained dynamics, and they are not experimental tests of the ChronaQ survivor-lineage theorem. A distinctive CQ1 experiment would require a conventional conditioned-quantum baseline, a ChronaQ admissible range, an observable renderer, an uncertainty law, a preregistered decision rule, and a consequence for a null result. No anomalous rate or dimensional prediction is asserted here.

10. Interpretation and downstream interfaces

10.1 The order established by CQ1

CQ1 establishes an information-refinement order on a qualifying finite record family. It is directed because a later accumulated record preserves its earlier prefix while possibly resolving an additional distinction. Structural volume says how many such prefixes remain; Shannon entropy says how expected information is distributed; τ supplies a convenient coordinate over a positive span.

Calling this result an "arrow of time" is acceptable only with that qualification. The theorem does not show that microscopic laws are irreversible or that entropy must be low at one cosmological end. Section 8 selects among candidate rereadings only relative to one declared record interaction and anchor algebra; it does not select the physical record family from every possible coarse-graining. Nor does it prevent differently situated observers from using refinements oriented relative to different boundaries. The physical content begins when stable records accessible to observers instantiate the abstract refinement.

The final boundary need not create a final tick. It can determine a completion, alter the positive law, or exclude candidate histories. Within a fixed admissible carrier it cannot make a previously realized prefix lack a later continuation. This is why boundary selection and record refinement must not be interleaved silently.

10.2 P2 and P3: order before geometry and observable compatibility

P2 may inherit the CQ1 order as an index for comparing compatible geometric structures. It may not define a metric simply by taking the reciprocal of S_0, S_1, mutual information, or branch probability. Metricity, causal structure, signature, dimensional scale, noise stability, and continuum behaviour require separate gates. A4 motivates that programme; P1 does not complete it.

P3 may use the record order to organize positive observational contexts. Its compatible observable families can preserve the same prefix and survivor constraints while remaining non-singleton. CQ1 therefore does not solve the latent selector problem. A unique positive result in one context does not imply a unique global signed completion, and naturality or cocycle conditions across contexts must be checked separately.

10.3 P5.5: finite bridge constraints

P5.5 inherits the requirement that any proposed finite bridge preserve declared resolutions and survivor relations. If it maps between latent and observable descriptions, it must send later record values to the correct earlier prefixes and preserve positivity where probabilities are reported. Existence, closedness, continuity, and cocycle compatibility remain bridge obligations.

A numerical selector such as Kullback–Leibler divergence, total variation, or a hybrid objective is not made uniquely physical by CQ1. Such a selector can be a useful chosen coordinate or solver objective, but its authority must come from a later axiom, theorem, or discriminator rather than from entropy monotonicity alone.

10.4 Rendered action work and P4

Rendered finite action calculations may use the declared Shannon τ as their standard dimensionless ordering coordinate. This permission is narrow. It means that a downstream calculation can index fields, links, or finite differences by the same governed chart rather than inventing an inconsistent time label.

P4 must independently construct every local object it needs. CQ1 supplies no local entropy density, entropy current, production term, stress-energy tensor, conserved source, coframe, connection, Ward identity, Bianchi identity, continuum measure, Newton constant, or Einstein dynamics. A finite difference with respect to τ becomes part of a physical law only after the action, transformation properties, normalization, and continuum interpretation are supplied. The existing gravitational law, if derivable from ChronaQ, must emerge through that independent action bridge rather than by rebranding the CQ1 monotone.

10.5 The role of a standard coordinate

Adopting Shannon τ reduces ambiguity without eliminating it. It gives later papers a common convention for a finite positive record family, much as a coordinate choice makes calculations comparable without becoming an invariant observable itself. Any conclusion invariant under monotone reparameterization should be identified as an order result. Any conclusion that depends on the exact Shannon spacing must disclose that dependence.

This distinction provides a useful diagnostic. If a proposed downstream effect disappears under a small admissible change of entropy order, it is not a consequence of survivor order alone. It may still be a consequence of the chosen Shannon chart plus additional dynamics, but the manuscript must say so. The CQ1 hierarchy therefore improves both mathematical clarity and experimental design.

11. Limitations, falsifiers, and open obligations

11.1 Finite and exact scope

The proof concerns finite supports. It does not establish a measure on an infinite path space, convergence under arbitrary refinements, semicontinuity of entropy, or a continuum history law. Infinite cardinalities can remain equal under strict refinement, while differential and relative entropies introduce dependence on reference measures. A continuum extension would need a specified measurable space, a consistent family of finite-dimensional distributions, an appropriate entropy functional, and a theorem controlling the limiting process.

Record membership is also exact. Approximate decoherence, noisy discrimination, and thresholded supports can make Hartley entropy discontinuous: an arbitrarily small positive weight changes the support count. Shannon entropy is more stable under small changes on a fixed finite alphabet, but its operational use still requires uncertainty bounds and a declared treatment of unresolved or censored alternatives. Neither theorem licenses choosing a threshold after examining whether monotonicity holds.

11.2 Boundary existence and selection

P1 does not prove that every physically interesting pair of boundary conditions yields a nonempty carrier. Nor does it derive a unique final boundary or a selector among several compatible latent completions. A boundary pair with no admissible history is a substantive negative result. A boundary pair whose record family has zero Shannon span supplies no standard clock coordinate even if its histories are well-defined.

The parity condition in the six-record model is a transparent finite boundary, not a cosmological proposal. A physical application must motivate its boundary class independently of the desired entropy result and must report excluded, empty, and deterministic members. Otherwise two-boundary conditioning becomes an unconstrained way to select a preferred answer.

11.3 Falsifiers and failure gates

CQ1 is blocked or narrowed in a proposed domain if any of the following occurs.

- A rendered earlier lineage has no later history compatible with both fixed boundaries.
- A fine record cell cannot be assigned one earlier prefix without merging previously distinct records.
- The purported positive law fails normalization, refinement balance, or its declared decoherence tolerance.
- A proposed continuation passes diagonal entropy tests but fails record repeatability relative to its declared earlier record algebra.
- A claimed clock-bearing interval has zero Shannon span.
- The ordering changes under operationally indistinguishable threshold changes without an uncertainty analysis.
- A path-surprisal increment is treated as a local density or rate without a local measure and balance law.
- A physical-time statement lacks an independent clock mechanism and dimensional calibration.

These are application failures or domain exclusions, not reasons to alter the theorem after value opening. An honest negative result identifies which interface failed: admissibility, record refinement, positive rendering, coordinate eligibility, or physical calibration.

11.4 Open mathematical and physical work

The main mathematical obligations are a measurable-history extension, robustness bounds for approximate records, compatibility of local refinements across overlapping observational contexts, and a clear relationship between non-singleton latent families and positive record exports. The main physical obligations are a dynamical model of record formation and stability, an action or generator that respects the declared coordinate, a causal and geometric renderer, and discriminators against conventional conditioned quantum models.

Thermodynamic interpretation is another separate project. To connect CQ1 to a second-law statement one would need a macrostate algebra, conserved or slowly varying quantities, a physically justified coarse-graining, and an account of why the relevant records are robust for typical observers. The fact that Shannon record entropy grows under accumulated refinement is compatible with such a programme but does not complete it.

12. Conclusion

ChronaQ begins with a timeless complete description, two-boundary admissibility, positive observable rendering, and a later geometry interface. In P1, A1–A3 combine with finite genuine record refinement to produce CQ1. The image of the admissible carrier under each accumulated-record map defines the survivor lineages. Truncation is surjective, so survivor volume and Hartley entropy cannot decrease. A positive consistent prefix law then makes accumulated Shannon entropy nondecreasing by the chain rule.

The two monotones have different roles. Hartley entropy is the logarithm of the structural survivor volume. Shannon entropy is a weighted record monotone and, over a positive span, defines the chosen standard dimensionless coordinate τ. Rényi entropies show that monotonicity alone does not select that coordinate. Shannon is selected because it is continuous, additive by conditional increments, interpretable as expected surprisal, and already provides the common downstream rendered chart.

The exact six-record law demonstrates strict refinements, nonuniform Shannon increments, a boundary-determined plateau, and a kinematic clock-labelled embedding. The trace-split control shows why selected log probability cannot replace survivor volume: mass divides while record support grows. The separate record-family calculation shows why classical refinement is necessary but not sufficient: five positive diagonal tables pass support and Shannon tests, but A3 decoherence and repeatability retain only the stable pointer lineage relative to the declared earlier record. Its phase-invariant weights and exact $\ln(2)$ increment then survive four frozen future boundaries without refitting, while the complementary eraser response remains phase-sensitive.

CQ1 is deliberately finite and conditional, but it is sufficient to anchor the next stages of ChronaQ. Later papers can inherit its order and declared Shannon chart while remaining responsible for observable compatibility, geometry, action, continuum structure, and experimental discrimination. The finite quantum gate also identifies a concrete route from abstract partitions to physically stable record families without claiming a universal basis. Together these results turn an intuitive arrow into a theorem with explicit premises and leave the unsolved physics visible.

Appendix A. Proof details and entropy-family proposition

A.1 Partition formulation

Each record map r_k induces a partition $\mathcal{P}_k$ of Ω_{B_-,B_+} into nonempty fibres. The factorization $r_k=t_k\circ r_{k+1}$ is equivalent to saying that $\mathcal{P}_{k+1}$ refines $\mathcal{P}_k$: every fine cell lies inside exactly one coarse cell. The number of nonempty cells cannot decrease. Several fine cells may share one coarse cell, so truncation is generally many-to-one. What is forbidden is one fine cell straddling two coarse cells.

The partition statement also shows that support growth depends on the record algebra, not directly on the number of microhistories. Refining an unobserved microhistory description without creating a new stable record does not increase V_k. Conversely, one stable binary record can split many microhistory fibres while increasing survivor volume by only the number of nonempty rendered values.

A.2 Changing-carrier formulation

Suppose a calculation uses a level-dependent carrier Ω_k rather than one fixed Ω_{B_-,B_+}. Let $L_k=r_k(\Omega_k)$. A truncation map on L_{k+1} is insufficient by itself: one must also prove $t_k(L_{k+1})=L_k$. Without this continuation equality, support can fall because re-filtering has removed an earlier lineage.

This formulation separates splitting from pruning. Refinement can add distinctions, while a changed admissibility rule can remove them. CQ1 applies when the complete boundary constraints are fixed in advance or when an equivalent continuation theorem is supplied. A level-dependent numerical solver must therefore record the carrier identity and not infer continuation merely from matching labels.

A.3 Weighted formulation

Let $Y=Z_{\leq k+1}$ have positive law p_{k+1}, and let $X=t_k(Y)=Z_{\leq k}$. The coarser law is $p_k(x)=\Sigma_{y:t\_k(y)=x}p_{k+1}(y)$. Shannon's chain rule gives $H(Y)=H(X)+H(Y|\ X)$, proving Theorem 2. If laws at successive levels are estimated or fitted independently and do not satisfy this pushforward relation, the comparison is not governed by the theorem.

For a path, $p_{k+1}(y)=p_k(x)p(y|\ x)$. Taking minus the logarithm turns the product into a sum. The increment is nonnegative because $0<p(y|\ x)\leq 1$. A zero-probability path has infinite or undefined surprisal and is not part of the positive rendered support.

A.4 Proposition: classical Rényi entropy under refinement

Proposition. Let Y be a finite positive record and $X=f(Y)$ a deterministic coarse-graining. For every $\alpha\in[0,\infty]$, $H_\alpha(Y)\geq H_\alpha(X)$.

Proof. For $\alpha=0$, the map from the support of Y onto the support of X is surjective, so the support cardinality cannot increase under coarse-graining. For $\alpha=1$, the statement is Shannon's chain rule. For $0<\alpha<1$, concavity of the alpha-power function implies that replacing child probabilities by each parent sum reduces the sum of alpha powers; division by $1-\alpha>0$ preserves the inequality. For $1<\alpha<\infty$, convexity makes the parent power sum at least the child power sum, while $1-\alpha<0$ reverses the logarithmic ratio to the same entropy ordering. For $\alpha=\infty$, the largest parent probability is at least the largest child probability, hence min-entropy cannot be greater after coarse-graining. ■

The proposition assumes a deterministic coarse-graining of one positive law. It does not establish monotonicity for quantum Rényi entropies under arbitrary channels, for independently chosen distributions, or for signed latent objects.

Appendix B. Reproducibility and data availability

The finite survivor, trace-split, six-record, Rényi-family, prefix-balance, normalized-coordinate, first-order Markov diagnostic, kinematic history-state, record-family selection, and held-out boundary checks are deterministic. The reproducibility supplement accompanying this preprint contains standalone calculation entry points, exact contracts, source manifests with SHA-256 identities, machine-readable result packets, plotting code, and focused tests. It records the complete admissible histories and probabilities, tolerance policy, candidate-selection values, held-out boundary rows, all numerical residuals quoted in the text, and the absence of fitted physical targets. No empirical participant data or proprietary experimental dataset was used.

The supplement is distributed with the preprint materials. Correspondence concerning the manuscript or reproducibility packet should be addressed to Stephen McIntyre at stephen.mcintyre@mcintyrelab.org.

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